OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS **4726**

Further Pure Mathematics 2

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- List of Formulae (MF1)

Other Materials Required:
- Scientific or graphical calculator

Thursday 27 May 2010
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Turn over

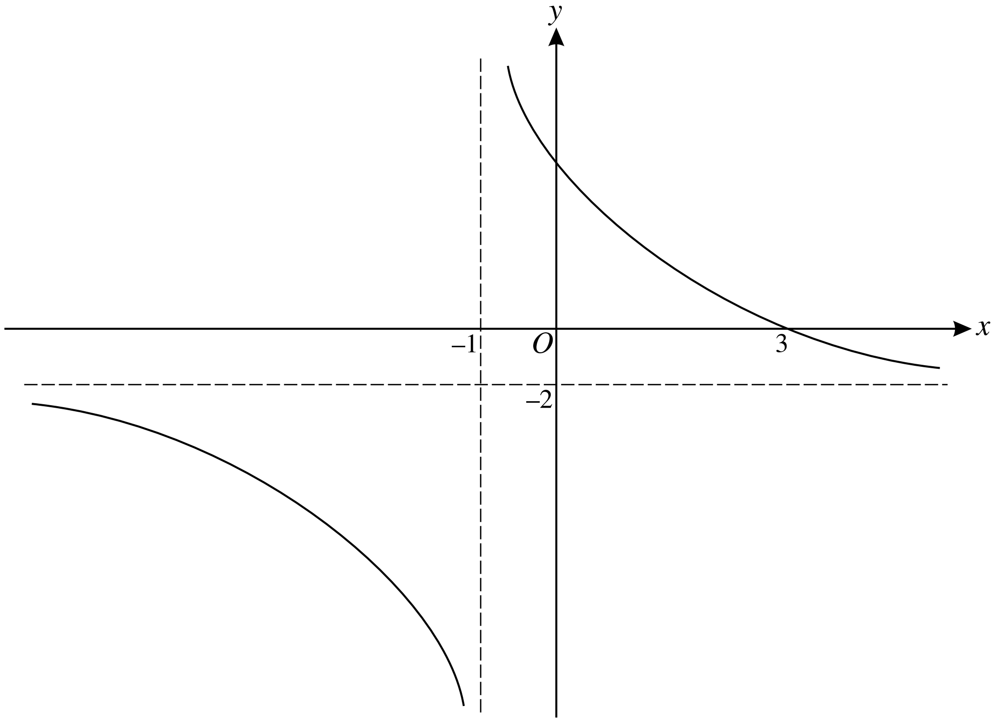
1 It is given that $f(x) = \tan^{-1} 2x$ and $g(x) = p \tan^{-1} x$, where p is a constant. Find the value of p for which $f'(\frac{1}{2}) = g'(\frac{1}{2})$. **[4]**

2 Given that the first three terms of the Maclaurin series for $(1 + \sin x)e^{2x}$ are identical to the first three terms of the binomial series for $(1 + ax)^n$, find the values of the constants a and n. (You may use appropriate results given in the List of Formulae (MF1).) **[6]**

3 Use the substitution $t = \tan \frac{1}{2}x$ to show that

$$\int_0^{\frac{1}{3}\pi} \frac{1}{1 - \sin x}\, dx = 1 + \sqrt{3}.$$ **[6]**

4



The diagram shows the curve with equation

$$y = \frac{ax + b}{x + c},$$

where a, b and c are constants.

(i) Given that the asymptotes of the curve are $x = -1$ and $y = -2$ and that the curve passes through $(3, 0)$, find the values of a, b and c. **[3]**

(ii) Sketch the curve with equation

$$y^2 = \frac{ax + b}{x + c},$$

for the values of a, b and c found in part **(i)**. State the coordinates of any points where the curve crosses the axes, and give the equations of any asymptotes. **[4]**

5 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \, dx.$$

(i) Prove that, for $n \geqslant 1$,

$$I_n = 2nI_{n-1} - 1.$$ **[4]**

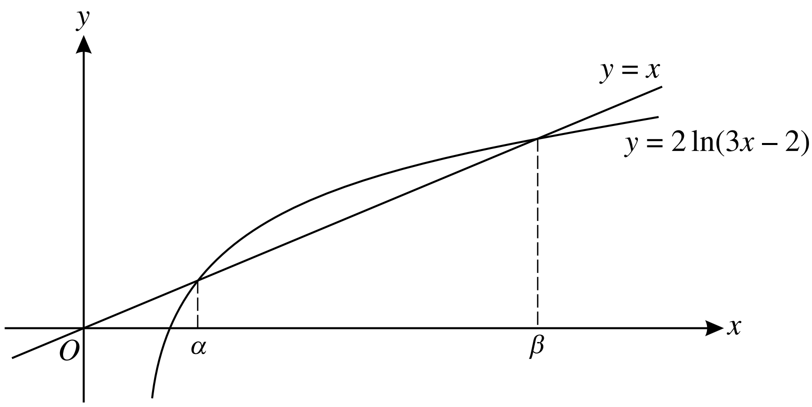
(ii) Find the exact value of I_3. **[4]**

6 **(i)** Show that $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$. **[2]**

(ii) Given that $y = \cosh(a \sinh^{-1} x)$, where a is a constant, show that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - a^2y = 0.$$ **[5]**

7



The line $y = x$ and the curve $y = 2\ln(3x-2)$ meet where $x = \alpha$ and $x = \beta$, as shown in the diagram.

(i) Use the iteration $x_{n+1} = 2\ln(3x_n - 2)$, with initial value $x_1 = 5.25$, to find the value of β correct to 2 decimal places. Show all your working. **[2]**

(ii) With the help of a 'staircase' diagram, explain why this iteration will not converge to α, whatever value of x_1 (other than α) is used. **[3]**

(iii) Show that the equation $x = 2\ln(3x-2)$ can be rewritten as $x = \frac{1}{3}\left(e^{\frac{1}{2}x} + 2\right)$. Use the Newton-Raphson method, with $f(x) = \frac{1}{3}\left(e^{\frac{1}{2}x} + 2\right) - x$ and $x_1 = 1.2$, to find α correct to 2 decimal places. Show all your working. **[4]**

(iv) Given that $x_1 = \ln 36$, explain why the Newton-Raphson method would not converge to a root of $f(x) = 0$. **[2]**

[Questions 8 and 9 are printed overleaf.]

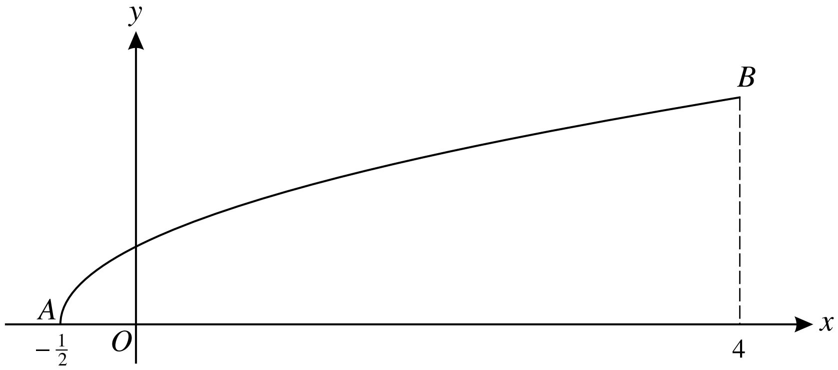
8 **(i)** Using the definition of $\cosh x$ in terms of e^x and e^{-x}, show that

$$4\cosh^3 x - 3\cosh x \equiv \cosh 3x.$$ **[4]**

(ii) Use the substitution $u = \cosh x$ to find, in terms of $5^{\frac{1}{3}}$, the real root of the equation

$$20u^3 - 15u - 13 = 0.$$ **[6]**

9



The diagram shows the curve with equation $y = \sqrt{2x+1}$ between the points $A\,(-\frac{1}{2}, 0)$ and $B\,(4, 3)$.

(i) Find the area of the region bounded by the curve, the x-axis and the line $x = 4$. Hence find the area of the region bounded by the curve and the lines OA and OB, where O is the origin. **[4]**

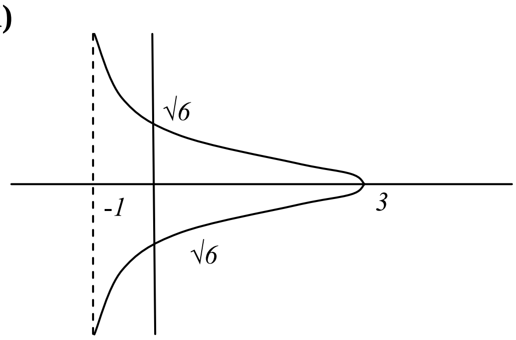
(ii) Show that the curve between B and A can be expressed in polar coordinates as

$$r = \frac{1}{1-\cos\theta}, \quad \text{where } \tan^{-1}\left(\tfrac{3}{4}\right) \leqslant \theta \leqslant \pi.$$ **[5]**

(iii) Deduce from parts **(i)** and **(ii)** that $\displaystyle\int_{\tan^{-1}\left(\frac{3}{4}\right)}^{\pi} \operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right) d\theta = 24.$ **[4]**

1	Derive/quote $g'(x)=p/(1+x^2)$	B1
	Attempt $f'(x)$ as $a/(1+bx^2)$	M1 Allow any a, b=2 or 4
	Use $x=½$ to set up a solvable equation in p, leading to at least one solution	M1
	Get $p={}^{5}/_{4}$ only	A1 AEEF
2	Reasonable attempt at $e^{2x}(1+2x+2x^2)$	M1 3 terms of the form $1+2x+ax^2$, $a\neq 0$
	Multiply out their expressions to get all terms up to x^2	M1 (3 terms) x (minimum of 2 terms)
	Get $1+3x+4x^2$	A1 cao
	Use binomial, equate coefficients to get 2 solvable equations in a and n	M1 Reasonable attempt at binomial, each term involving a and n ($an=3$, $a^2n(n-1)/2=4$)
	Reasonable attempt to eliminate a or n	M1
	Get $n=9$, $a=⅓$ cwo	A1 cao
		SC Reasonable $f'(x)$ and $f''(x)$ using product rule (2 terms) M1
		Use their expressions to find $f'(0)$ and $f''(0)$ M1
		Get $1+3x+4x^2$ cao A1
3	Quote/derive correct $dx=2dt/(1+t^2)$	B1
	Replace all x (not $dx=dt$)	M1 From their expressions
	Get $2/(t-1)^2$ or equivalent	A1
	Reasonable attempt to integrate their expression	M1
	Use correct limits in their correct integral	A1√ Must involve √3
	Clearly tidy to √3+1 from cwo	A1 A.G.
4 (i)	Get $a=-2$	B1 May be quoted (from correct working)
	Get $b=6$	B1 May be quoted (from correct working)
	Get $c=1$	B1 May be quoted (from correct working)

(ii)



B1 Correct shape in $-1<x\leq 3$ only (allow just top or bottom half)

B1 90^0 (at x=3) (must cross x-axis i.e. symmetry)

B1 Asymptote at $x=-1$ only (allow −1 seen)

B1√ Correct crossing points; $\pm\sqrt{(b/c)}$ from their b,c

5 (i) Reasonable attempt at parts	M1	Leading to second integral
Get $e^x(1-2x)^n - \int e^x.n(1-2x)^{n-1}.-2\,dx$	A1	Or $(1-2x)^{n+1}/(-2(n+1))e^x - \int (1-2x)^{n+1}/(-2(n+1))e^x dx$
Evidence of limits used in integrated part	M1	Should show ±1
Tidy to A.G.	A1	Allow $I_{n+1} = 2(n+1)I_n - 1$
(ii) Show any one of $I_3=6I_2-1$, $I_2=4I_1-1$, $I_1=2I_0-1$	B1	May be implied
Get $I_0(=e^{½}-1)$ or $I_1(=2e^{½}-3)$	B1	
Substitute their values back for their I_3	M1	Not involving n
Get $48e^{½} - 79$	A1	
6 (i) Reasonable attempt to differentiate $\sinh y = x$ to get dy/dx in terms of y	M1	Allow $\pm\cosh y\, dy/dx = 1$
Replace $\sinh y$ to A.G.	A1	Clearly use $\cosh^2 - \sinh^2 = 1$ SC Attempt to diff. $y = \ln(x+\sqrt{(x^2+1)})$ using chain rule M1 Clearly tidy to A.G. A1
(ii) Reasonable attempt at chain rule	M1	To give a product
Get $dy/dx = a \sinh(a\sinh^{-1}x)/\sqrt{(x^2+1)}$	A1	
Reasonable attempt at product/quotient	M1	Must involve sinh and cosh
Get d^2y/dx^2 correctly in some form	A1√	From $dy/dx = k \sinh(a\sinh^{-1}x)/\sqrt{(x^2+1)}$
Substitute in and clearly get A.G.	A1	SC Write $\sqrt{(x^2+1)}dy/dx = k \sinh(a\sinh^{-1}x)$ or similar Derive the A.G.
7 (i) Get 5.242, 5.239, 5.237	B1√	Any 3(minimum) correct from previous value
Get 5.24	B1	Allow one B1 for 5.24 seen if 2 d.p.used
(ii) Show reasonable staircase for any region	B1	Drawn curve to line
Describe any one of the three cases	B1	
Describe all three cases	B1	
(iii) Reasonable attempt to use log/expo. rules	M1	Allow derivation either way
Clearly get A.G.	A1	
Attempt $f'(x)$ and use at least once in correct N-R formula	M1	
Get answers that lead to 1.31	A1	Minimum of 2 answers; allow truncation/rounding to at least 3 d.p.
(iv) Show $f'(\ln 36) = 0$	B1	
Explain why N-R would not work	B1	Tangent parallel to Ox would not meet Ox again or divide by 0 gives an error

8	**(i)** Use correct definition of cosh x	B1	
	Attempt to cube their definition involving e^x and e^{-x} (or $e^{2x}$and e^x)	M1	Must be 4 terms
	Put their 4 terms into LHS and attempt to simplify	M1	
	Clearly get A.G.	A1	
			SC Allow one B1 for correct derivation from $\cosh 3x = \cosh(2x+x)$
	(ii) Rewrite as $k\cosh 3x = 13$	M1	
	Use ln equivalent on $13/k$	M1	Allow ±ln or $\ln(13/k \pm\sqrt{(13/k)^2 - 1})$ for their k or attempt to set up and solve quadratic via exponentials
	Get $x = (\pm)\ ⅓\ln 5$	A1	
	Replace in cosh x for u	M1	
	Use $e^{a\ln b} = b^a$ at least once	M1	
	Get $½(5^{⅓} + 5^{-⅓})$	A1	
9	**(i)** Attempt integral as $k(2x+1)^{1.5}$	M1	
	Get 9	A1	cao
	Attempt subtraction of areas	M1	Their answer – triangle
	Get 3	A1√	Their answer − 6 (>0)
	(ii) Use $r^2=x^2+y^2$ and $x=r\cos\theta$, $y=r\sin\theta$	B1	
	Eliminate x and y to produce quadratic equation (=0) in r (or $\cos\theta$)	M1	
	Solve their quadratic to get r in terms of θ (or vice versa)	A1√	
	Clearly get A.G.	A1	r>0 may be assumed
	Clearly show θ_1(at B)=$\tan^{-1}¾$ and θ_2 (at A) =π	B1	
			SC Eliminate y to get r in terms of x only M1 Get $r = x + 1$ A1 SC Start with r=1/(1-cosθ) and derive cartesian
	(iii) Use area = $½\int r^2\, d\theta$ with correct r	B1	cwo; ignore limits
	Rewrite as $k\,\text{cosec}^4(½\theta)$	M1	Not just quoted
	Equate to their part **(i)** and tidy	M1	To get ∫ = some constant
	Get 24	A1	A.G.